The theory of the jet-flap for unsteady motion

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(Received 12 September 1960)

Linearized methods developed in earlier papers (Spence 1956, 1961) are used to discuss the non-stationary flow of a wing with a jet-flap. We consider a thin two-dimensional wing at zero incidence in a steady stream of speed U, with a thin jet emerging parallel to the chord at the trailing edge, and study the motion following an instantaneous deflexion of the jet through an angle τ_0. If the momentum-flux coefficient C_J of the jet is $\leqslant 4$, the governing equations can be put in a form in which C_J does not appear explicitly, and a similarity solution then gives the shape of the jet at small times t from the start of the motion as a function of $(x-c)\mu^{-\frac{1}{3}}t^{-\frac{2}{3}}$, where $x=c$ is the trailing edge and $\mu=\frac{1}{4}C_J$. The solution is obtained from a certain third-order integro-differential equation, by constructing the Mellin transform of the non-dimensional shape. When t is large the jet near the wing approaches the shape given by the known results for steady flow, but its shape at distances of the order of Ut downstream changes diffusively under the action of the starting vortex. A similarity solution is also found for the flow in this region in terms of $(x-Ut)\mu^{-\frac{1}{3}}t^{-\frac{2}{3}}$, without restriction to small μ. Expressions for the lift coefficient at small and large times are found, and the case of an oscillating deflexion angle is treated by the same methods.

1. Introduction

In a paper published some years ago (Spence 1956; this will be referred to as I) the author used the methods of thin aerofoil theory to discuss the flow past a two-dimensional wing with a jet emerging from the trailing edge at a small angular deflexion τ relative to the chordline, into fluid moving with undisturbed speed U and constant density ρ. This represents the mathematical idealization of a jet blowing over a small flap. Numerical solutions were given in the paper for the lift derivatives with respect to the wing-incidence α and to the jet-deflexion τ as functions of the jet momentum coefficient C_J; more recently (Spence 1961, referred to as II) these have been replaced, using an analytic solution of the singular integro-differential equation for the slope of the jet, by expansions in powers of C_J and $\ln C_J$ valid for $\frac{1}{4}C_J$ ($=\mu$ say) less than 1—which range of momentum coefficients covers most practical applications.

In the present paper attention is turned to the unsteady motion of such a wing, and in particular to that which would follow a time-dependent change in the flap-angle τ. This problem can be formulated in linearized approximation using the model introduced in I, and the analytic methods of II can be used to obtain

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closed solutions of the resulting flow problem in a number of limiting cases. We restrict ourselves here to motions in which the wing remains at zero incidence, and τ is a prescribed function of the time t; the same methods could, however, be used to discuss more general types of motion—for instance, flight through a sharp-edged gust, or following a sudden change of incidence. For such motions the theory of aerofoils without jets is well developed, and a synthesis of the results found here with those of the classical unsteady aerofoil theory will be given later by J. C. Erickson.

The problem is formulated in § 2 in terms of a distribution of vorticity representing the wing and jet. In linearized approximation this is located in the plane of the wing, and suitable boundary conditions are applied on the normal component of the velocity induced by it at this plane. In § 2.4 the lift coefficient in unsteady flow is expressed as the time-derivative of a certain integral of the stated vortex distribution, to be evaluated for particular cases in later sections. The derivation depends on the fact that the total vorticity inside a sufficiently large contour enclosing the aerofoil remains zero for all time if it is so initially, positive vorticity on and near the wing being balanced by an equal amount of opposite sign which is swept back at stream speed. This negative distribution is analogous to the starting vortex of classical theory, but differs in that it spreads diffusively over a growing region as a result of changes induced in the curvature of the jet. The constancy of circulation around a large contour cannot be deduced in the usual way from Kelvin's theorem, since in the presence of a jet extending to infinity a large circuit surrounding the wing and moving with the fluid could deform in such a way as not to close across the jet; but nevertheless it follows from the governing equations, as is shown in § 2.3.

Two particular types of time-dependence have been examined in the remainder of the paper: first, that of a sudden change in deflexion, described by

$$\tau(t) = \begin{cases} 0 & (t \leqslant 0), \\ \tau_0 & (t > 0), \end{cases} \tag{1}$$

(a non-zero value for $t \leqslant 0$ would be taken care of by the steady solution of II) and second, steady oscillation with reduced frequency n, i.e.

$$\tau(t) = \tau_0 \exp(inUt/c), \tag{2}$$

where c is the chord length. The second is closely related to the operational solution of the first and is therefore treated rather briefly in § 5. The remarks just made about the constancy of circulation in a large circuit do not of course apply in this case unless the motion is assumed to have been in existence only for a finite time. From the solution of the first case, that for an arbitrary transient $\tau(t)$ can be written down as a convolution integral, so these two are sufficient for a fairly full discussion of the problem. The solution for the sudden-deflexion case is developed in detail in §§ 3 and 4 for small and large times, respectively. For small times, the transformation used in II to introduce the jet-strength parameter $\mu = \frac{1}{4}C_J$ into the co-ordinates in such a way as to magnify the trailing edge region results in a great simplification of the equations, which then possess a similarity solution in terms of $x/t^{\frac{2}{3}}$ when μ is small. With the aid of a similar transformation of the co-ordinates in the neighbourhood of the point $x = Ut$ it is also possible to discuss the flow at large times, in this case without restriction on the value of μ, and again a 'similarity' solution is found for the downwash distribution over the jet far from the wing.

2. Formulation of the problem

We suppose as before that the jet is a thin two-dimensional curved sheet of high velocity gas in irrotational motion, with time-dependent streamline boundaries as in figure 1, and are again able to write the pressure difference Δp between points immediately above and immediately below the jet, and lying on the same normal to the internal streamlines, as the product of the mean local curvature κ and the jet momentum-flux J.

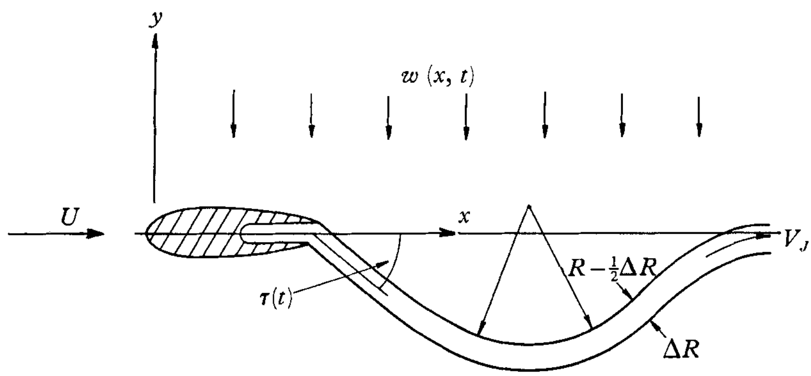

FIGURE 1. Flow in an element of the jet (schematic).

2.1. *Pressure rise across a thin jet*

This property, first pointed out by Erickson (1959), follows from the fact that the normal is an equipotential of the flow within the jet, along which $p + \frac{1}{2}\rho_J V^2$ is therefore constant at any instant, where p, V and ρ_J are the pressure, velocity and density within the jet. Hence to the first order in small quantities

$$\Delta p + \rho_J \bar{V} \Delta V = 0,$$

where Δ represents the difference between the values of a quantity on the upper and lower boundaries, and $\bar{V}$ is the mean value of V across the jet. The condition of irrotationality within the jet may likewise be expressed as

$$(\Delta V/\Delta R) + (\bar{V}/R) = 0,$$

where $R \mp \frac{1}{2}\Delta R$ are the radii of curvature of upper and lower boundaries. Hence, eliminating ΔV,

$$\Delta p = -(\rho_J \bar{V}^2 \Delta R)/R = -\kappa J, \tag{3}$$

and in the limit $\Delta R \to 0$, an extension of the argument used in I again shows that J is constant along the jet.

2.2. *Linearized form of equations*

The pressure difference given by (3) may be set equal to that in the outside stream, in which the velocity potential is $\phi(x, y, t)$ say. Again we linearize the problem by applying the boundary conditions on the line $y = 0$, and using the unsteady form of Bernoulli's equation, are able to write

$$\rho\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\phi+\Delta p = 0, \tag{4}$$

where $$\Delta\phi(x,t) = \phi(x,+0,t)-\phi(x,-0,t).$$

Replacing Δp in this equation by (3), and writing the curvature as $-\partial^2 h_0/\partial x^2$, where $h_0(x,t)$ is the displacement of the jet below the real axis, we then have

$$x > c: \quad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\phi = -\tfrac{1}{2}cC_J U^2\frac{\partial^2 h_0}{\partial x^2}, \tag{5}$$

the momentum-flux coefficient C_J being defined as $J/\frac{1}{2}\rho U^2 c$. If now the wing and jet are represented by a distribution of vorticity on the x-axis, of magnitude $\gamma(x,t) = (\partial/\partial x)\Delta\phi$, then the downward velocity $w(x,t) = -\phi_y(x,0,t)$ at the axis is related to γ as before by

$$w(x,t) = -\frac{1}{2\pi}\int_0^\infty \frac{\gamma(\xi,t)\,d\xi}{\xi-x}, \tag{6}$$

but the condition that the flow should be tangential to the jet becomes

$$x>c: \quad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)h_0(x,t) = w(x,t), \tag{7}$$

where $x = 0$ and $x = c$ represent the leading and trailing edges of the wing. An equation of the same form as the last must also be satisfied at the wing, $h_0(x,t)$ then being a prescribed function describing the applied motion. But as stated in the introduction, the present paper is restricted to consideration of cases in which the wing is at zero incidence to the stream, and only the jet-deflexion is time-dependent. Accordingly, we set

$$0 < x < c: \quad w(x,t) = 0 \tag{8}$$

on the wing, and specify the initial slope of the jet by means of

$$w(c,t) = U\left(\frac{\partial h_0}{\partial x}\right)_{x=c} = U\tau(t). \tag{9}$$

The first of these boundary conditions can be absorbed into the integral relating downwash to vorticity exactly as in the steady case, II § 2.2 (this form of reference to sections in II will be used throughout), so that (6) is again replaced by

$$x > c: \quad w(x,t) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{\frac{1}{2}}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{\frac{1}{2}}\frac{\gamma(\xi,t)\,d\xi}{\xi-x}, \tag{10}$$

and the problem is now specified on the interval $c < x < \infty$ by the last two equations together with (5) and (7). The latter may be combined, by cross-differentiation to eliminate h and $\Delta\phi$, into a single auxiliary equation between w and γ, namely

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\gamma(x,t) = -\tfrac{1}{2}cC_J U^2\left(\frac{\partial}{\partial x}\right)^3 w(x,t), \tag{11}$$

so that (10) and (11) are now the governing equations on $c < x < \infty$, subject to the remaining boundary condition (9). To find the position of the jet, as is done for small times in § 3, it is also necessary to specify its initial position by means of $h(x, 0) = 0$. Information about this is lostin the differentiation leading to (11), so this equation is not used in the short-time discussion; it is, however, the more useful form for treating the motion after times long enough for it to have become independent of the precise starting conditions.

2.3. *Circulation round a contour far from the wing*

In showing that $\Delta\phi(x,t) \to 0$ as $x \to \infty$ for all time, it will be assumed for the present that sufficiently far from the wing the jet returns to its undisturbed position along the axis, so that with increasing x the curvature finally vanishes.

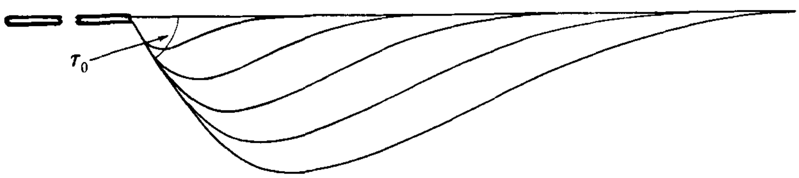

FIGURE 2. Successive jet shapes at small times after initial deflexion (equal intervals of $t^{\frac{3}{2}}$) (schematic).

Intuitively this seems clear, for if the final position were displaced vertically an infinite amount of work would have been done to the fluid in finite time; moreover, the assumption will be confirmed from the solutions for the jet shape to be found in later sections. To prove the result concerning $\Delta\phi$ we may solve equation (4) in terms of Δp in the form

$$\Delta\phi(x,t) = -(1/\rho U)\int_{\max(0,\, x-Ut)}^{x} \Delta p\left(\xi, t+\frac{\xi-x}{U}\right)d\xi + F(x-Ut),$$

where F is arbitrary. Since this solution holds for all positive or negative x and t (although the integrand vanishes when either of its arguments is less than zero) the initial conditions show that $F \equiv 0$. Then if $x - Ut > c$ we can replace Δp in the integrand by the jet curvature, by (3), obtaining

$$\Delta\phi(x,t) = -(J/\rho U)\int_{x-Ut}^{x} \kappa\left(\xi, t+\frac{\xi-x}{U}\right)d\xi.$$

If $$\mathrm{K}(x,t) = \max\left|\kappa\left(\xi, t+\frac{\xi-x}{U}\right)\right| \quad \text{for} \quad x-Ut \leqslant \xi \leqslant x,$$

the absolute value of the integral is $\leqslant (J/\rho U)\, Ut\mathrm{K}(x,t)$, and, for a fixed t,

$$\mathrm{K}(x,t) \to 0 \quad \text{as} \quad x \to \infty.$$

Therefore as stated $$\lim_{x\to\infty} \Delta\phi(x,t) = 0,$$

the limit being approached non-uniformly with respect to t—i.e. as indicated in figure 2, the larger the time, the further from the wing must one go before $|\Delta\phi(x,t)|$ becomes less than a chosen small quantity.

The result can be written in a number of equivalent ways. Using (6) and (10),

$$0 = \Delta\phi(\infty, t) = \int_0^\infty \gamma(\xi, t)\, d\xi = \lim_{x\to\infty} 2\pi x w(x, t) = \int_c^\infty \left(\frac{\xi}{\xi - c}\right)^{\frac{1}{2}} \gamma(\xi, t)\, d\xi. \quad (12)$$

It may be noted in passing that the last result in (12) is equivalent to the so-called 'Wagner condition' of classical unsteady aerofoil theory, when with a suddenly applied downwash $U\alpha$ at the wing the integral instead of vanishing would equal $-\pi U\alpha$ for all time. In the absence of a jet γ is a function of $Ut - x$ only, and is readily obtained in terms of Hankel functions on inversion of this integral condition (von Kármán & Sears 1938).

2.4. *Lift coefficient in unsteady flow*

The lift force L on the wing is the sum of the integrated normal presures over the chord and the direct component $J \sin \tau$ of the jet momentum at exit. Writing τ for $\sin \tau$,

$$L = -\int_0^c \Delta p\, dx + J\tau(t) = \rho \frac{\partial}{\partial t}\int_0^c \Delta\phi\, dx + \rho U \Delta\phi(c, t) + J\tau(t), \quad (13)$$

using (4) for the pressure difference at the wing. The last term on the right is ρ times the integral of the right-hand side of (5) from $x = c$ to $x = \infty$, so (13) can be replaced by the single integral

$$C_L = L/\tfrac{1}{2}\rho U^2 c = (2/U^2 c)\int_0^\infty dx \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\phi \quad (14)$$

for the lift coefficient. In the transient case in which $\Delta\phi(\infty, t)$ is zero, the second term makes no contribution, leaving

$$C_L = (2/U^2 c)\frac{\partial}{\partial t}\int_0^\infty \Delta\phi\, dx = -(2/U^2 c)\frac{\partial}{\partial t}\int_0^\infty x\gamma(x, t)\, dx \quad (15)$$

on integration by parts. $\int_0^\infty \Delta\phi\, dx$ may be identified as ρ^{-1} times the 'impulse' of the system. Since the integral equation for $\gamma(x, t)$ has been formulated on the interval $c < x < \infty$, it is convenient to eliminate the integration from 0 to c in (15). To do so, use equation (7) of II, to write the vorticity distribution on the wing in terms of that on the jet as

$$0 \leqslant x \leqslant c: \quad \gamma(x, t) = \frac{1}{\pi}\left(\frac{c - x}{x}\right)^{\frac{1}{2}} \int_c^\infty \left(\frac{\xi}{\xi - c}\right)^{\frac{1}{2}} \frac{\gamma(\xi, t)\, d\xi}{\xi - x}. \quad (16)$$

Then
$$-\int_0^c x\gamma(x, t)\, dx = \int_c^\infty \left(\frac{\xi}{\xi - c}\right)^{\frac{1}{2}} \gamma(\xi, t) \left[\frac{1}{\pi}\int_0^c \frac{x^{\frac{1}{2}}(c - x)^{\frac{1}{2}}}{x - \xi}\, dx\right] d\xi$$

and from a well-known result (see, for example, I, equation 66) the integral in square brackets is found to equal $[\xi^{\frac{1}{2}}(\xi - c)^{\frac{1}{2}} - \xi + \frac{1}{2}c]$; combining this with $\int_c^\infty \xi\gamma(\xi, t)\, d\xi$ and making use of (12) enables one finally to write the lift coefficient as

$$C_L = -(2/U^2 c)\frac{\partial}{\partial t}\int_c^\infty \xi^{\frac{1}{2}}(\xi - c)^{\frac{1}{2}} \gamma(\xi, t)\, d\xi. \quad (17)$$

3. Sudden change in deflexion: small-time solution

In this section a solution valid for small times will be found for the first case mentioned in the introduction, that in which the wing is at zero incidence throughout, and at $t = 0$ the angular deflexion of the jet at the trailing edge changes instantaneously from 0 to τ_0. The motion in the region $x > c$ is then described by the equations of § 2.2, subject to the boundary conditions

$$h_0(c,t) = h_0(x,0) = 0, \quad \left(\frac{\partial h_0}{\partial x}\right)_{x=c} = \tau_0. \tag{18}$$

Physically one might expect the jet to deform somewhat in the manner indicated in figure 2, and a similarity solution showing this behaviour can in fact be found for its shape near the trailing edge. As a preliminary step the governing equations may be simplified, provided the jet-strength parameter $\mu = \frac{1}{4}C_J$ is sufficiently small, less than $\frac{1}{4}$ say—this is not a severe restriction on C_J—by absorbing μ into the co-ordinates near the trailing edge. As in II, write

$$\left.\begin{aligned} h_0(x,t) &= \mu c\tau_0(x/c)^{-\frac{1}{2}}\, h(x',t'), \\ \gamma(x,t) &= 2U\tau_0(x/c)^{-\frac{1}{2}}\, g(x',t'), \\ \text{where} \qquad x' &= \frac{x-c}{\mu c}, \quad t' = \frac{Ut}{\mu c}, \end{aligned}\right\} \tag{19}$$

and excluding terms of order μ, equations (5), (7) and (10) become

$$\frac{\partial h}{\partial t'} + \frac{\partial h}{\partial x'} = -\frac{1}{\pi}\int_0^\infty \left(\frac{x'}{\xi'}\right)^{\frac{1}{2}} \frac{g(\xi',t')\,d\xi'}{\xi'-x'}, \tag{20}$$

$$\frac{\partial g}{\partial t'} + \frac{\partial g}{\partial x'} = -\frac{\partial^3 h}{\partial x'^3}, \tag{21}$$

with boundary conditions derived from (18)

$$h(x',0) = h(0,t') = 0, \quad h_{x'}(0,t') = 1. \tag{22}$$

If $t' \ll 1$ an approximation to these equations may be made by omitting the derivatives with respect to x' on the left-hand side of (20) and (21), since they are smaller than those with respect to t' except at points so close to the trailing edge that $x'/t' = O(1)$. Cross-differentiation of the approximate equations gives

$$h_{t't'} = \frac{1}{\pi}\int_0^\infty \left(\frac{x'}{\xi'}\right)^{\frac{1}{2}} \frac{h_{\xi'\xi'\xi'}\,d\xi'}{\xi'-x'}. \tag{23}$$

A similarity solution for (23), satisfying the boundary conditions (22), is found by writing

$$h(x',t') = t'^{\frac{2}{3}} f(x'/t'^{\frac{2}{3}}) = t'^{\frac{2}{3}} f(z) \tag{24}$$

say, when the equation becomes

$$f - zf' - 2z^2 f'' = -(9/2\pi)\int_0^\infty \left(\frac{z}{\zeta}\right)^{\frac{1}{2}} \frac{f'''(\zeta)\,d\zeta}{\zeta - z}. \tag{25}$$

The first boundary condition is automatically satisfied and the remaining two become

$$f(0) = 0, \quad f'(0) = 1. \tag{26}$$

The solution for this equation is found in § 3.1. The existence of the solution justifies *a posteriori* the omission of the x'-derivatives from (20) and (21), since

$$(\partial/\partial x')/(\partial/\partial t') \ll O(t'^{\frac{1}{3}}) \ll 1$$

for small t'. The solution is not, however, uniformly valid, but applies only when $x'/t' \ll 1$; when the reverse inequality holds the time derivatives disappear from the equations, leaving the equation for steady motion that was solved in II. The latter therefore correctly describes the singular behaviour of the vortex distribution very close to the trailing edge, with validity in a neighbourhood that grows linearly with time.

Equation (23) can also be derived in another way which displays its region of validity more clearly. If $t = t'$ and $y = x' - t'$, say, are treated as the independent variables, $\partial/\partial t' + \partial/\partial x'$ is replaced by $\partial/\partial t$, and $\partial/\partial x'$ by $\partial/\partial y$, so that (20) and (21) become

$$\frac{\partial h}{\partial t} = -\frac{1}{\pi}\int_{-t}^{\infty}\left(\frac{y+t}{\eta+t}\right)^{\frac{1}{2}}\frac{g(\eta,t)\,d\eta}{\eta-y}, \quad \frac{\partial g}{\partial t} = -\frac{\partial^3 h}{\partial y^3}.$$

For small t the first of these can be approximated provided $y/t \gg 1$ by setting $t = 0$ in the kernel and at the lower limit, after which (23) is recovered on cross-differentiation.

3.1. *Similarity solution for the jet-shape*

Equation (25) may be solved by the method used by Lighthill (1959)—see also II § 3.2—for the basic equation of steady motion. Write

$$f(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}F(s)\,ds \tag{27}$$

for some real c, where $F(s)$ is the Mellin transform $\int_0^\infty z^{s-1}f(z)\,dz$. Differentiation gives

$$-\tfrac{2}{9}(f - zf' - 2z^2f'') = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2s+2}{3}\right)\left(\frac{2s-1}{3}\right)z^{-s}F(s)\,ds,$$

$$f''' = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s-3}s(s+1)(s+2)F(s)\,ds.$$

If $s(s+1)(s+2)F(s)$ is regular in the strip $c-3 < \mathscr{R}s < c$, the contour in the last integral can be changed to $\mathscr{R}s = c-3$, whence on writing $s-3$ for s the integral becomes

$$f''' = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}(s-3)(s-2)(s-1)F(s-3)\,ds. \tag{28}$$

Then, using the result $\displaystyle\frac{1}{\pi}\int_0^\infty\left(\frac{z}{\zeta}\right)^{\frac{1}{2}}\frac{\zeta^{-s}}{\zeta-z}d\zeta = -\tan\pi s$

which holds for $|\mathscr{R}s| < \frac{1}{2}$ to evaluate the Cauchy integral in (25), we obtain the functional equation

$$\left(\frac{2s+2}{3}\right)\left(\frac{2s-1}{3}\right)F(s) = (s-3)(s-2)(s-1)(\tan\pi s)F(s-3), \tag{29}$$

provided $|c| < \frac{1}{2}$. To simplify the equation write $F(s) = P(s)\,Q(s)$ where

$$P(s) = \frac{(s-1)!}{\frac{1}{3}(2s+2)!}\frac{\left(\sin\frac{\pi s}{3}\right)\left(\sin\pi\frac{s-1}{3}\right)}{\left(\sin\pi\frac{s-\frac{1}{2}}{3}\right)}, \tag{30}$$

when it becomes

$$Q(s) = -(\tan\pi s)\,Q(s-3) = \left(\tan\pi\frac{s-1}{3}\right)\left(\tan\pi\frac{s}{3}\right)\left(\tan\pi\frac{s+1}{3}\right)Q(s-3). \tag{31}$$

Clearly one solution of (31) is

$$G_0\left(\frac{s-1}{3}\right)G_0\left(\frac{s}{3}\right)G_0\left(\frac{s+1}{3}\right) = \Psi(s), \quad \text{say}, \tag{32}$$

where $G_0(s)$ is the Lighthill function defined in II (equation 38), which satisfies

$$G_0(s) = (\tan\pi s)\,G_0(s-1), \quad G_0(0) = G_0(-\tfrac{1}{2}) = 1, \tag{33}$$

and has poles of order n at $s = -n$ and at $s = n-\frac{1}{2}$, where n is a positive integer.

The general solution must be $\Psi(s)$ multiplied by a function of period 3, and to decide whether any such further factor is necessary we must see whether the conditions imposed on $F(s)$ in deriving the functional equation (29) are satisfied. The function $P(s)$ has a simple pole at $s = -1$ with residue $\frac{3}{4}$ and is otherwise regular in $\mathscr{R}s < \frac{1}{2}$, the remaining poles being those of $\operatorname{cosec}\frac{1}{3}\pi(s-\frac{1}{2})$ at $s = 3n+\frac{1}{2}$ ($n = 0, 1, 2, \ldots$). The three factors of $\Psi(s)$ have simple poles at $s = -2, -3, -4$ and at $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$, respectively. Therefore if we choose $0 < c < \frac{1}{2}$,

$$s(s+1)(s+2)\,P(s)\,\Psi(s)$$

is regular as required in the strip $c-3 < \mathscr{R}s < c$.

Finally, $F(s)$ must be integrable as $|\mathscr{I}s| \to \infty$ on $\mathscr{R}s = c$ in order for the integral (27) to exist. It was shown in II § 3.3 that $G_0(s) \sim \exp(-\frac{1}{2}\pi|\mathscr{I}s|)$, so by (31)

$$\Psi(s) \sim \exp(-\tfrac{1}{2}\pi|\mathscr{I}s|).$$

Also

$$P(s) \sim \exp(\tfrac{1}{6}\pi|\mathscr{I}s|).$$

The product of these factors $\sim \exp(-\frac{1}{3}\pi|\mathscr{I}s|)$, and the inclusion of a further periodic factor would either introduce poles into the strip $c-3 < \mathscr{R}s < c$ or prevent the convergence of $F(s)$ as $|\mathscr{I}s| \to \infty$. Therefore the solution is completed by writing

$$Q(s) = A\Psi(s), \tag{34}$$

where A is a constant to be determined from the second boundary condition (26). (The first has been satisfied by choosing $P(s)$ regular at $s = 0$.) Since

$$f'(z) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sF(s)\,z^{-s-1}\,ds,$$

the first term in the expansion of $f'(z)$ for small z is the residue of $-sF(s)$ at its pole at $s = -1$. The boundary condition is therefore satisfied if $F(s)$ has residue 1 at this point, where $P(s)$ has a simple pole with residue $\frac{3}{4}$ and $\Psi(s)$ is regular. Therefore

$$A = \frac{4}{3\Psi(-1)} = \tfrac{4}{3}[G_0(-\tfrac{2}{3})\,G_0(-\tfrac{1}{3})]^{-1}. \tag{35}$$

3.2. *Lift coefficient*

Inserting the co-ordinate transformation (19) into the expression (17) for the lift coefficient, we obtain

$$C_L = -4\tau_0\mu^{\frac{1}{2}}\frac{\partial}{\partial t'}\int_0^\infty \xi^{\frac{1}{2}}g(\xi,t')\,d\xi.$$

This may be evaluated from the solution of the last section, since $\partial g/\partial t' = -\partial^3 h/\partial x'^3$ in the small-time approximation; thus in terms of the similarity variable defined by (24)

$$C_L = 4\tau_0\mu^{\frac{1}{2}}t'^{-\frac{1}{3}}\int_0^\infty z^{\frac{1}{2}}f'''(z)\,dz. \tag{36}$$

To evaluate this expression invert (28), obtaining

$$\int_0^\infty z^{s-1}f'''(z)\,dz = -(s-3)(s-2)(s-1)F(s-3).$$

The required integral is the value of the right-hand side when $s = 3$, namely

$$-\frac{(\frac{1}{2})!\,(-\sin\frac{1}{2}\pi)(-\sin\frac{5}{6}\pi)}{(-\frac{1}{3})!\,(-\sin\frac{2}{3}\pi)}\frac{G_0(-\frac{5}{6})\,G_0(-\frac{1}{6})}{(\frac{3}{4})\,G_0(-\frac{2}{3})\,G_0(-\frac{1}{3})}.$$

Since $G_0(s) = G_0(-\frac{1}{2}-s)$, (by II, equation 35), $G_0(-\frac{1}{6}) = G_0(-\frac{1}{3})$ and

$$G_0(-\tfrac{5}{6}) = (\cot\tfrac{1}{6}\pi)\,G_0(\tfrac{1}{6}) = \sqrt{3}\,G_0(-\tfrac{2}{3}),$$

so that finally

$$C_L = \frac{8\tau_0(\pi\mu)^{\frac{1}{2}}}{3(-\frac{1}{3})!}t'^{-\frac{1}{3}} = 3{\cdot}4905\tau_0\mu^{\frac{5}{6}}\left(\frac{Ut}{c}\right)^{-\frac{1}{3}}. \tag{37}$$

Thus in a sudden deflexion the lift force is infinite initially. Physically this implies that an infinite rate of working is required to start the motion impulsively. If the deflexion were described more generally by some function $\tau(t)$ the lift coefficient for small times could be written as a Stieltjes integral

$$C_L = 3{\cdot}4905\mu^{\frac{5}{6}}\left(\frac{U}{c}\right)^{-\frac{1}{3}}\int_0^{\tau(t)}(t-u)^{-\frac{1}{3}}\,d\tau(u),$$

showing that infinite values of C_L occur only if $d\tau/dt$ is singular at $t = 0$, as is the case with the delta-function representation of equation (1).

3.3. *Jet shape*

The shape of the jet for small and large values of z is found by moving the contour $\mathscr{R}s = c$ in (27) over successive poles of the integrand. The residues are evaluated in the same way as in II § 3.4 using the properties of the G_0 function listed there. Those at the poles $s = -n$, where n is a positive integer, give the expansion for $z < 1$, and those at $s = n - \frac{1}{2}$ that for $z > 1$. The double pole at $s = \frac{1}{2}$ produces a logarithmic leading term in the asymptotic expansions

$$f(z) \sim \begin{cases} z - \dfrac{2z^2}{3^{\frac{3}{2}}(\frac{1}{3})!} - \dfrac{2z^3}{27(-\frac{1}{3})!} + O(z^4\ln z) & \text{for} \quad z < 1, \\ \dfrac{1}{(\pi z)^{\frac{1}{2}}}\left[\dfrac{3}{\pi}\ln(4z) + C - \dfrac{3}{5(-\frac{1}{3})!}\dfrac{1}{z} - \dfrac{3}{28(\frac{1}{3})!}\dfrac{1}{z^2}\right] + O\left(\dfrac{\ln z}{z^3}\right) & \text{for} \quad z > 1. \end{cases} \tag{38}$$

where $$C = \frac{\gamma+3}{\pi} - \frac{4}{3^{\frac{3}{2}}} = 0{\cdot}3689,$$

γ being Euler's constant. The function $f(z)$ is plotted in figure 3, the neighbourhood of $z = 1$ having been supplied by eye, and plots of

$$h(x', t') = t'^{\frac{2}{3}} f(x'/t'^{\frac{2}{3}})$$

confirm that the shape of the jet for small times is as indicated in figure 2. In physical co-ordinates the leading terms are

$$h(x,t) = \begin{cases} x' + O(x'^2/t'^{\frac{2}{3}}) & \text{for} \quad x < t^{\frac{2}{3}}, \\ \dfrac{t'}{(\pi x')^{\frac{1}{2}}}\left(\dfrac{3}{\pi}\ln x'\right) & \text{for} \quad x > t^{\frac{2}{3}}, \end{cases} \tag{39}$$

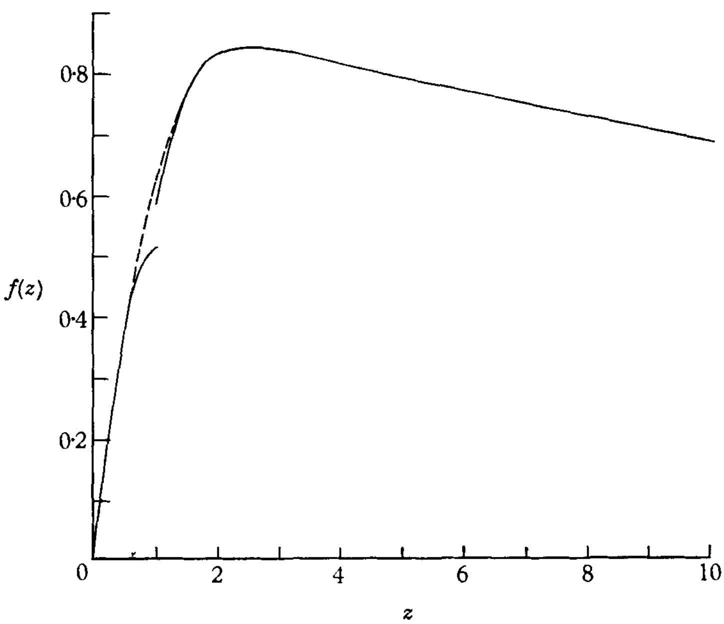

FIGURE 3. Non-dimensional shape-function $f(z)$ for small times.

according to (38), but in fact, for $x' \ll t'$, the steady solution of II in which $h(x) = x + O(x^2 \ln x)$ should apply; the term of order $x'^2/t'^{\frac{2}{3}}$ in (39) is valid only when $t' \ll x' \ll t'^{\frac{2}{3}}$.

4. Sudden change in deflexion: large-time solution

When the time from the instant of jet deflexion is long, the flow in the neighbourhood of the wing approaches that described by the steady solution of II. The integral of the vorticity on the wing and in its neighbourhood therefore approaches the value $\Gamma_\infty = \frac{1}{2}UcC_L^{(\infty)}$ say, where $C_L^{(\infty)}$ is the lift coefficient in steady flow with the final deflexion. But since the total circulation about the system, $\Delta\phi(\infty, t)$, is zero initially, it remains so for all time, a balance being brought about by negative vorticity which is swept downstream and causes the jet far from the wing to deform as indicated in figure 4. After a sufficiently long time, this negative vorticity acts on the wing as would a line vortex of magnitude $-\Gamma_\infty$ at a distance Ut downstream, producing a downwash $\Gamma_\infty/(2\pi Ut)$ at the wing. This is equivalent to a reduction in incidence of amount $(c/4\pi Ut)\, C_L^{(\infty)}$. In the opposite direction, at distances large compared with Ut, the combined distribution is seen as a doublet of strength $-\Gamma_\infty Ut$ at the origin.

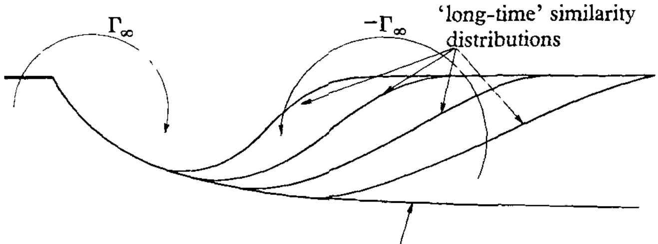

FIGURE 4. Successive jet-shapes at long times from start (schematic).

4.1. *Final form of the jet far from the wing*

To discuss the flow mathematically, we may write the distributions of vorticity and downwash as

$$\left.\begin{aligned} \gamma(x,t) &= \gamma_\infty(x)+\gamma_1(y,t),\\ w(x,t) &= w_\infty(x)+w_1(y,t),\\ \text{where}\qquad y &= x-Ut, \end{aligned}\right\} \tag{40}$$

$\gamma_\infty(x)$ and $w_\infty(x)$ being the distributions in steady flow with deflexion τ_0. The Wagner condition $\Delta\phi(\infty,t)=0$ is

$$0=\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{\frac{1}{2}}\gamma(\xi,t)\,d\xi=\Gamma_\infty+\int_{c-Ut}^\infty\left(\frac{\eta+Ut}{\eta+Ut-c}\right)^{\frac{1}{2}}\gamma_1(\eta,t)\,d\eta$$

by (12), and subtracting the corresponding steady equations (II, equations 8 and 5) from (10) and (11), the equations connecting w_1 and γ_1 are obtained as

$$w_1(y,t)=-\frac{1}{2\pi}\left(\frac{y+Ut-c}{y+Ut}\right)^{\frac{1}{2}}\int_{c-Ut}^\infty\left(\frac{\eta+Ut}{\eta+Ut-c}\right)^{\frac{1}{2}}\frac{\gamma_1(\eta,t)\,d\eta}{\eta-y}$$

and
$$\frac{\partial^2\gamma_1}{\partial t^2}=-\tfrac{1}{2}cC_J\,U^2\frac{\partial^3 w_1}{\partial y^3}. \tag{41}$$

From these it appears that for large t, γ_1 and w_1 approach the functions $\gamma_0(y,t)$ and $w_0(y,t)$ say, which satisfy the equations obtained by allowing t to tend to infinity wherever it occurs explicitly in the last three equations, namely

$$\int_{-\infty}^\infty \gamma_0(\eta,t)\,d\eta=-\Gamma_\infty \tag{42}$$

and
$$w_0(y,t)=-\frac{1}{2\pi}\int_{-\infty}^\infty\frac{\gamma_0(\eta,t)\,d\eta}{\eta-y}, \tag{43}$$

respectively, together with (41) which is unaffected. Note that two separate steps have been made in deriving (43): first, the kernel

$$\left(\frac{y+Ut-c}{y+Ut}\frac{\eta+Ut}{\eta+Ut-c}\right)^{\frac{1}{2}}$$

has been set equal to unity, which is equivalent to disregarding the boundary condition $w_1(y,t) = 0$, $-Ut < y < -Ut+c$, obtained from (8), and is justifiable provided $w_0(y,t)$ is small enough on this interval for large t; secondly, the lower limit of integration has been changed from the trailing edge $y = -Ut+c$ to $y = -\infty$, which is justifiable provided $\gamma_0(\eta,t)$ dies out sufficiently rapidly for large negative η. The solution found below is shown in § 4.3 to meet both these requirements.

4.2. *Similarity solutions for the downwash distribution*

The last three equations can be solved for γ_0 and w_0 in terms of a similarity parameter proportional to $y/t^{\frac{2}{3}}$. As in the small time solution the jet strength parameter $\mu = \frac{1}{4}C_J$ can be adsorbed into the co-ordinate system, but by contrast with that case there is no need here to restrict μ to small values, since the transformed equations are exactly independent of μ.

Write

$$\left.\begin{aligned} y &= \mu c y', \quad t = \frac{\mu c t'}{U} \\ \text{and set}\qquad \gamma_0(y,t) &= \frac{\Gamma_\infty}{\mu c}t'^{-\frac{2}{3}}g(z), \\ w_0(y,t) &= \frac{\Gamma_\infty}{2\mu c}t'^{-\frac{2}{3}}f(z), \\ \text{where}\qquad z &= y'/t'^{\frac{2}{3}}. \end{aligned}\right\} \tag{44}$$

Then equations (41) to (43) become

$$\tfrac{10}{9}g + 2zg' + \tfrac{4}{9}z^2g'' = -f''', \tag{45}$$

$$\int_{-\infty}^{\infty} g(\zeta)\,d\zeta = -1, \tag{46}$$

and

$$f(z) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\zeta)\,d\zeta}{\zeta-z}. \tag{47}$$

To solve these equations by the Mellin transform technique used earlier, we first change to the semi-infinite interval $0 < z < \infty$ by defining for $z > 0$

$$\left.\begin{aligned} f_1(z) &= \tfrac{1}{2}f(z)-\tfrac{1}{2}f(-z), \\ f_2(z) &= \tfrac{1}{2}f(z)+\tfrac{1}{2}f(-z), \\ g_1(z) &= \tfrac{1}{2}g(z)+\tfrac{1}{2}g(-z), \\ g_2(z) &= \tfrac{1}{2}g(z)-\tfrac{1}{2}g(-z). \end{aligned}\right\} \tag{48}$$

Then, for $z > 0$, f_1 is related to g_1, and f_2 to g_2 by (45), and the two remaining equations become

$$\int_0^\infty g_1(\zeta)\,d\zeta = -\tfrac{1}{2}, \quad \int_0^\infty g_2(\zeta)\,d\zeta \quad \text{exists,} \tag{49}$$

and

$$f_1(z) = -\frac{2z}{\pi}\int_0^\infty \frac{g_1(\zeta)\,d\zeta}{\zeta^2 - z^2}, \tag{50}$$

$$f_2(z) = -\frac{2}{\pi}\int_0^\infty \frac{\zeta g_2(\zeta)\,d\zeta}{\zeta^2 - z^2}. \tag{51}$$

The solution of (50) is carried out in the Appendix, and it is also shown that the only solution of (51) such that $\int_0^\infty g_2(\zeta)\,d\zeta$ exists as required by (49) is the trivial solution

$$f_2(z) \equiv g_2(z) \equiv 0. \tag{52}$$

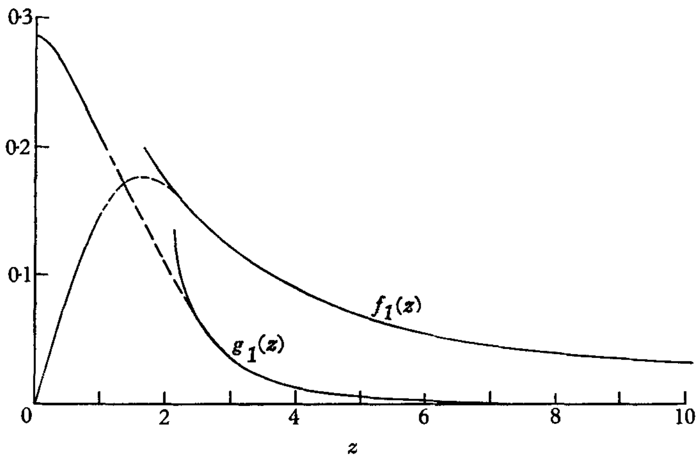

FIGURE 5. Distributions of downwash (f_1) and vorticity (g_1) from long-time solution.

Therefore, for $z > 0$,

$$\left.\begin{aligned} f(z) &= -f(-z) = f_1(z), \\ g(z) &= g(-z) = g_1(z), \end{aligned}\right\} \tag{53}$$

and the solutions for $f_1(z)$, $g_1(z)$ found in the Appendix are

$$\left.\begin{aligned} z<1:\quad f_1(z) &= -\frac{2}{3\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4n+1}{3}\right)!\,\frac{z^{2n+1}}{(2n+1)!}, \\ g_1(z) &= -\frac{2}{3\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4n-1}{3}\right)!\,\frac{z^{2n}}{(2n)!}, \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} z>1:\quad f_1(z) &\sim -\frac{1}{\pi z}\left\{1+\frac{3}{4z}\sqrt{\frac{\pi}{2z}}+\frac{315}{64z^3}\sqrt{\frac{\pi}{2z}}+\dots\right\}, \\ g_1(z) &\sim -\frac{3}{4z^2\sqrt{2\pi z}}\left\{1+\frac{4}{z\sqrt{2\pi z}}-\frac{315}{48z^3}+\dots\right\}. \end{aligned}\right\} \tag{55}$$

The two latter are the leading terms of asymptotic expansions in which the full series are divergent for all $z \geqslant 1$. These functions are plotted in figure 5.

4.3. *Limiting forms of the downwash distribution*

It is easy to confirm from (55) that the downwash at distances $x \gg Ut$ from the wing has, as expected, the distribution produced by a doublet $-\Gamma_\infty Ut$ at the origin, since for large positive values of $y = x - Ut$, from (44) and (55)

$$w_0 = \frac{\Gamma_\infty}{2\mu c} t'^{-\frac{2}{3}} f\left(\frac{y'}{t'^{\frac{2}{3}}}\right) \sim -\frac{\Gamma_\infty}{2\pi\mu c y'} = -\frac{\Gamma_\infty}{2\pi(x-Ut)},$$

whereas $w_\infty \sim \Gamma_\infty/2\pi x$, so that altogether

$$x \gg Ut: \quad w(x,t) \sim \frac{\Gamma_\infty}{2\pi}\left(\frac{1}{x} - \frac{1}{x-Ut}\right) = -\frac{\Gamma_\infty Ut}{2\pi x^2} + O\left(\frac{t^2}{x^3}\right). \tag{56}$$

The leading term is that produced by the stated doublet. The equation also shows that the jet shape sufficiently far from the wing will be undisturbed at any given time, thereby justifying the argument used in § 2.3 to show that

$$\lim_{x\to\infty} \Delta\phi(x,t) = 0.$$

Close to the wing, on the other hand, $y = x - Ut \sim -Ut$, in which case we find in the same way

$$x \ll Ut: \quad w_0 \sim \frac{\Gamma_\infty}{2\pi Ut}. \tag{57}$$

The last two results could of course have been written down at once on the physical grounds mentioned at the beginning of § 4, but the mathematical discussion is unavoidable if one is to seek higher approximations in $1/t$ to the flow near the wing. The fact that the downwash due to γ_0 dies out at the wing like $1/t$ justifies the first step in forming equation (43), and the justification for the second step follows from the fact that the contribution from the tail of the γ_0 distribution, namely

$$\int_{-\infty}^{-Ut} \frac{\gamma_0(\eta,t)\,d\eta}{\eta - y},$$

behaves like

$$-\frac{3\Gamma_\infty(\mu c)^{\frac{1}{2}}\,Ut}{4(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{-Ut} \frac{d\eta}{(-\eta)^{\frac{5}{2}}(\eta - Ut + x)} = O\left[\left(\frac{Ut}{c}\right)^{-\frac{3}{2}} \ln\frac{Ut}{x}\right] \tag{58}$$

for large t (using the asymptotic expansion for $g(z)$), and consequently is an order of magnitude smaller than the leading term.

4.4. *Lift coefficient at large times*

To discuss the flow near the wing at times $t \gg c/U$ say, we may expand the time-dependent part $\gamma_1(y,t)$ of the full vorticity distribution, equation (40), in the form

$$\gamma_1(x-Ut,t) = \gamma(x,t) - \gamma_\infty(x) = \gamma_0(y,t) - \frac{c}{Ut}\gamma_2(x) + o\left(\frac{c}{Ut}\right) \tag{59}$$

say, for $x \ll Ut$. The corresponding downwash is

$$w_1 = w(x,t) - w_\infty(x) = w_0(y,t) + \frac{c}{Ut} w_2(x) + o\left(\frac{c}{Ut}\right)$$

say, where
$$w_2(x) = -\frac{1}{2\pi}\int_0^\infty \frac{\gamma_2(\xi)\,d\xi}{\xi - x}. \tag{60}$$

Then since both w_0, γ_0 and w_∞, γ_∞ satisfy (11) separately, on equating terms of order $1/t$ in that equation we find
$$\gamma_2(x) = -\tfrac{1}{2}cC_J w_2'(x). \tag{61}$$

The condition that the downwash w_1 vanishes over the wing (where $w_\infty(x)$ is also zero) is, to order $1/t$,
$$0 \leqslant x \leqslant c: \quad w_0 + \frac{c}{Ut}w_2 = \frac{c}{Ut}\left(\frac{\Gamma_\infty}{2\pi c} + w_2\right) = 0. \tag{62}$$

The last three equations show that $w_2(x)$ is precisely the distribution of downwash in steady flow past a wing at incidence $\alpha = -\Gamma_\infty/2\pi cU$ with a tangential (i.e. undeflected) jet at the trailing edge, and from the results for that problem (see I or II, or Thwaites 1960, pp. 500 *et seq.*) we can deduce in particular that
$$\int_0^c \gamma_2(\xi)\,d\xi = -\tfrac{1}{2}Uc\alpha\left(\frac{\partial C_L^{(\infty)}}{\partial\alpha} - C_J\right), \tag{63}$$

$\partial C_L^{(\infty)}/\partial\alpha$ being the lift derivative with respect to incidence for the value of C_J in question. From equation (13) it is seen that to order $1/t$ the lift coefficient differs from $C_L^{(\infty)}$ only by reason of the contribution of $\gamma_2(x)$ to $(\Delta\phi)_{x=c}$, which is c/Ut times (63); thus since
$$\alpha = -\Gamma_\infty/2\pi cU = -C_L^{(\infty)}/4\pi,$$
$$\frac{C_L}{C_L^{(\infty)}} = 1 - \frac{1}{4\pi}\left(\frac{\partial C_L^{(\infty)}}{\partial\alpha} - C_J\right)\left(\frac{c}{Ut}\right) + o\left(\frac{c}{Ut}\right). \tag{64}$$

For small values of C_J, $\partial C_L^{(\infty)}/\partial\alpha \to 2\pi$, so in the limit we recover the result
$$C_L/C_L^{(\infty)} = 1 - c/(2Ut) + \ldots$$
of the classical unsteady aerofoil theory for a sudden change of incidence.

From (58) it appears that the next term in the expansion will be of order $t^{-\frac{3}{2}}\ln t$; to calculate this however it would be necessary to derive an expression for $\gamma_0 - \gamma_1$ far from the wing and this has not been attempted. It is possible to show that $C_L/C_L^{(\infty)} = 1 + O(c/Ut)$ simply by inserting the limiting distribution $\gamma_0(y,t)$ in the integral (17), since the contribution from $\gamma_\infty(x)$ disappears on differentiation with respect to t, but the precise coefficient of $1/t$ cannot be found by this means.

5. Oscillating flap-angle

In this section the solution of the problem presented by a steady harmonic variation of the initial angle τ will be outlined. For simplicity it will be assumed that $\mu = \frac{1}{4}C_J$ is small enough to permit the co-ordinate-stretching transformation which renders the equations independent of μ to first order. We set
$$\tau(t) = \tau_0 e^{inUt/c} \tag{2}$$

and seek a solution
$$\left.\begin{aligned} h_0(x,t) &= \mu c\tau_0 e^{inUt/c}(x/c)^{-\frac{1}{2}}h(x'),\\ \gamma(x,t) &= 2U\tau_0 e^{inUt/c}(x/c)^{-\frac{1}{2}}g(x'),\end{aligned}\right\} \quad (65)$$
where as before
$$x' = \frac{x-c}{\mu c}.$$

Then excluding terms of order μ and omitting the primes, (5), (7) and (10) become

$$i\nu h + h' = -\frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}}\frac{g(\xi)\,d\xi}{\xi - x} \quad (66)$$

and
$$i\nu g + g' = -h''', \quad (67)$$

where $\nu = \mu n$, and the boundary conditions are

$$h(0) = 0, \quad h'(0) = 1. \quad (68)$$

Iterative solutions to these equations can be found in the cases $\nu \ll 1$, $\nu \gg 1$, which we now treat in turn.

5.1. *Low-frequency oscillation*

If $\nu < 1$, which is the case for small μ if the physical reduced frequency n is of order 1, the first approximation is found by setting $\nu = 0$, when the basic steady equation

$$Lh'(x) \equiv h'(x) - \frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}}\frac{h''(\xi)\,d\xi}{\xi - x} = 0, \quad h'(0) = 1 \quad (69)$$

studied in II is obtained. The solution will be denoted here by $h_\infty(x)$ so that in the notation of II

$$h'_\infty(x) = f_0(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}(s-1)!\,G_0(s)\,ds,$$

$G_0(s)$ being the Lighthill function defined by equation (33) of the present paper. Since $h_\infty(0) = 0$,

$$h_\infty(x) = \int_0^x f_0(\xi)\,d\xi. \quad (70)$$

Putting this back in (66) and (67), we can write

$$h(x) = h_\infty(x) + i\nu h_1(x) + \ldots, \quad (71)$$

where $h_1(x)$ satisfies $h_1(0) = h_1'(0) = 0$, and

$$Lh_1'(x) = -h_\infty(x) - \frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}}\frac{h'_\infty(\xi)\,d\xi}{\xi - x}. \quad (72)$$

By the methods of II § 3.5 the solution is found to be

$$h_1'(x) = -2\int_0^x \xi h''_\infty(\xi)\,d\xi = -2\int_0^x \xi f_0'(\xi)\,d\xi. \quad (73)$$

The first term in (71) represents a slow oscillation of the basic steady solution, and the second a small correction 90° out of phase; this will be modified in turn by higher terms. The solution is not valid for large x, since $h_\infty(x) \sim 2(x/\pi)^{\frac{1}{2}}$ and the term νh which is excluded in forming the first approximation dominates over that retained if we go far enough from the wing. Moreover it is physically unrealistic to imagine the jet oscillating with an amplitude that tends to infinity far downstream. In fact since the motion must have started at some large negative time $-T$, at distances of the order of UT from the wing the jet will be described by the long-time transient solution of § 4.

5.2. *High-frequency oscillation*

In the other extreme case, $\nu \gg 1$, a first approximation $h_0(x)$ say is obtained by omitting $h'(x)$ and $g'(x)$ from (66) and (67), respectively, leaving the equation

$$-\nu^2 h_0(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}} \frac{h_0'''(\xi)\,d\xi}{\xi - x}, \tag{74}$$

with boundary conditions given by (68). In the same way as the low-frequency oscillation is closely related to the long-time solution, this case is related to the small-time solution: in fact (74) differs from the equation satisfied by the Fourier transform of the jet-shape for small times only in the absence of a term $g_\xi(\xi, 0)$ from the numerator in the integrand, arising from the initial value of the vorticity distribution. The solution of (74) can be carried out by the method used in § 3, and we find

$$H_0(s) = \int_0^\infty x^{s-1} h_0(x)\,dx = -A\nu^{-\frac{2}{3}(s+1)}(s-1)!\left(\sin\frac{\pi s}{3}\right)\left(\sin\pi\frac{s-1}{3}\right)\Psi(s), \tag{75}$$

where $\Psi(s)$ is the product defined by (32) and A the constant $4/3\Psi(-1)$ given by (35). $h_0(x)$ is therefore $\nu^{-\frac{2}{3}}$ times a function of $\nu^{\frac{2}{3}}x$.

In this case the expansion can be continued by writing

$$\left.\begin{aligned} h(x) &= h_0(x) + i\nu^{-\frac{1}{3}} h_1(x) + \ldots, \\ g(x) &= g_0(x) + i\nu^{-\frac{1}{3}} g_1(x) + \ldots, \end{aligned}\right\} \tag{76}$$

where h_1 and g_1 are given by

$$-\nu^{\frac{2}{3}} h_1 + h_0' = -\frac{i\nu^{-\frac{1}{3}}}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}} \frac{g_1(\xi)\,d\xi}{\xi - x},$$

$$-\nu^{\frac{2}{3}} g_1 + g_0' = -i\nu^{-\frac{1}{3}} h_1'''(\xi).$$

These equations may be combined to eliminate g_1 in the form

$$\begin{aligned} \nu^2 h_1 + \frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}} \frac{h_1'''(\xi)\,d\xi}{\xi - x} &= \nu^{\frac{4}{3}} h_0' - \frac{\nu^{-\frac{2}{3}}}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{\frac{1}{2}} \frac{h_0^{(\mathrm{iv})}(\xi)\,d\xi}{\xi - x} \\ &= 2\nu^{\frac{4}{3}} h_0'(x). \end{aligned} \tag{77}$$

(The last step follows on differentiation of (74), since

$$\int_0^\infty \xi^{-\frac{1}{2}} h_0^{(\mathrm{iv})}(\xi)\,d\xi = (\tfrac{1}{2})!\,\Psi(-\tfrac{7}{2}) = 0.)$$

The solution of (77) can be found by the same methods; again we find a third-order difference equation for the Mellin transform

$$H_1(s) = \int_0^\infty x^{s-1} h_1(x)\,dx.$$

The equation is not homogeneous like (29), but has a term from the right-hand side of (77) involving $H_0(s-1)$, and after some analysis which will be omitted here, the solution obtained is

$$H_1(s) = \frac{2}{3}\left(\operatorname{cosec}\frac{\pi s}{3}\right)\left[(s+1)\left(\cos\pi\frac{s-2}{3}\right) - \frac{1}{\sqrt{3}}\left(\sin\pi\frac{s-2}{3}\right)\right] H_0(s). \quad (78)$$

5.3. *Lift coefficient with an oscillating jet*

Equation (17) which gives the lift coefficient in terms of the vorticity distribution on the jet depends on the fact that $\Delta\phi$ tends to zero far from the wing. This holds in a steady oscillation too, if it is assumed that the motion started at some large but finite negative time $-T$, in which case the flow remains undisturbed at distances from the wing that are large compared with UT. In the low-frequency case however the equation cannot be applied directly, since when the expression for the transformed vorticity $g(x')$ obtained from the analysis of §5.1 is inserted in (17) the integral in that equation is divergent. This is because of the invalidity of the solution for large x' that was mentioned earlier. To get round the difficulty we now obtain an expression for C_L in which the lowest order terms in μ can be found as finite integrals of $g(x')$. Equation (13) can be rearranged to give

$$C_L = \frac{2}{Uc}\int_0^c \gamma(\xi,t)\,d\xi + \frac{2}{U^2c}\frac{\partial}{\partial t}\int_0^c (c-\xi)\,\gamma(\xi,t)\,d\xi + C_J\tau(t).$$

Using the identity (16) to replace the integrals from 0 to c by integrals from c to ∞, we obtain

$$C_L = \frac{2}{Uc}\int_c^\infty\left\{\left(\frac{\xi}{\xi-c}\right)^{\frac{1}{2}} - 1\right\}\gamma(\xi,t)\,d\xi$$
$$+\frac{1}{U^2}\frac{\partial}{\partial t}\int_c^\infty\left\{\left(\frac{\xi}{\xi-c}\right)^{\frac{1}{2}} + 2\left(\frac{\xi-c}{c}\right) - 2\left(\frac{\xi}{c}\right)^{\frac{1}{2}}\left(\frac{\xi-c}{c}\right)^{\frac{1}{2}}\right\}\gamma(\xi,t)\,d\xi + C_J\tau(t). \quad (79)$$

Inserting the expression (65) for $\gamma(\xi,t)$, this becomes simply

$$C_L = 4\tau_0\mu^{\frac{1}{2}}\,e^{inUt/c}\left(1+\frac{in}{2}\right)\int_0^\infty \xi'^{-\frac{1}{2}}g(\xi')\,d\xi' + o(\mu^{\frac{1}{2}}). \quad (80)$$

If n is of order unity, ν is of order μ and the low frequency solution of §5.1 applies. $g(\xi)$ is equal to $f_0'(\xi)$ and the integral in (80) is therefore known from II to be $\pi^{\frac{1}{2}}$, whence

$$C_L = 4\tau_0(\pi\mu)^{\frac{1}{2}}\exp\left(inUt/c-\epsilon\right), \quad (81)$$

where the phase angle $\epsilon = \tan^{-1}(\frac{1}{2}n)$, showing that the lift coefficient oscillates with a phase lag increasing with frequency about the value $4\tau_0(\pi\mu)^{\frac{1}{2}}$ which corresponds to a steady deflexion τ_0. The effect of the higher-frequency terms will be to change both phase and amplitude by amounts of order ν.

In the high-frequency case, on the other hand, the difficulties with the integral in (17) do not arise. Substituting from (65), the expression for the lift coefficient is found to be

$$C_L = -4\mu^{\frac{1}{2}}\tau_0\,e^{inUt/c}\int_0^\infty \xi^{\frac{1}{2}}\,i\nu g(\xi)\,d\xi.$$

The first approximation to $i\nu g(\xi)$ is $-h_0'''(\xi)$, on insertion of which the integral may be evaluated as $\frac{4}{3}H_0(-\frac{3}{2})$, yielding the expression

$$C_L = 4\tau_0(\pi\mu/3)^{\frac{1}{2}}\nu^{\frac{1}{3}}e^{inUt/c}. \tag{82}$$

This shows the amplitude to be frequency-dependent in a way that fits neatly with the $t^{-\frac{1}{3}}$ dependence found in the related short-time discussion of § 3.2, and also that the phase-lag has decreased to zero.

6. Discussion

The analysis shows that a first approximation for small times to the flow with a suddenly deflected jet can be found in the 'similarity' form used in § 3. The same solution can also be found by writing equation (23) operationally in terms of the Laplace transform with respect to time. The author has not found a way of improving this solution by a higher term, as was done for the high-frequency oscillating case in § 5.2, on account of its invalidity near $x' = 0$. The conclusion that the lift-coefficient varies for small times as $t^{-\frac{1}{3}}$ would not be altered by a more exact analysis, since there is always a region to which the similarity solution applies. The lift coefficient comes entirely from the apparent mass terms in the limit, and it appears that the force required to deflect the jet impulsively must be infinite. The 'impulse' of the fluid $\int_0^t L\,dt$ is however finite (and zero initially).

It may be noted that the long-time solution for the motion far from the wing is independent of the precise way in which the flow was started, depending only on the total circulation in the final steady flow near the wing and therefore on the final deflexion of the jet. This solution would also apply if the wing were at incidence. The similarity solution for $\gamma_0(y,t)$ found in § 4 actually describes the way a jet extending from $y = -\infty$ to $y = +\infty$ would deform if at time $t = 0$ it lay wholly along the axis and a vortex of magnitude $-\Gamma_\infty$ was concentrated at the origin.

I am grateful to Professor W. R. Sears and Mr J. C. Erickson for calling my attention to the problem and for many helpful discussions, and also to Mr Erickson for verifying the analysis in the manuscript. This work was carried out under the sponsorship of the Mechanics Division, Office of Scientific Research, USAF (Contract AF 49(638)-544).

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Appendix. Integral equations for the downwash far from the wing

The equations derived in § 4 connecting f_1 (the anti-symmetric part of the downwash distribution) and g_1 (the symmetric part of the vorticity distribution) are

$$f_1(x) = -\frac{2x}{\pi}\int_0^\infty \frac{g_1(\xi)\,d\xi}{\xi^2-x^2} \tag{A 1}$$

and

$$\tfrac{2}{9}(5g_1+9xg_1'+2x^2g_1'') = -f_1''', \tag{A 2}$$

with the condition

$$\int_0^\infty g_1(\xi)\,d\xi = -\tfrac{1}{2}. \tag{A 3}$$

To solve we construct the Mellin transforms

$$F_1(s) = \int_0^\infty x^{s-1}f_1(x)\,dx, \quad G_1(s) = \int_0^\infty x^{s-1}g_1(x)\,dx, \tag{A 4}$$

from which on inversion f_1 and g_1 will be found as

$$f_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(s)\,x^{-s}ds, \quad g_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_1(s)\,x^{-s}ds, \tag{A 5}$$

where c is some real constant such that $x^{c-1}f_1(x)$, $x^{c-1}g_1(x) \in L(0,\infty)$. Since, provided $|\mathscr{R}s| < 1$,

$$-\frac{2x}{\pi}\int_0^\infty \frac{\xi^{-s}d\xi}{\xi^2-x^2} = x^{-s}\tan\frac{\pi s}{2}, \tag{A 6}$$

we obtain from (A 1), multiplying both sides by x^{s-1} and integrating from 0 to ∞,

$$F_1(s) = (\tan\tfrac{1}{2}\pi s)\,G_1(s), \tag{A 7}$$

provided $-1 < c < 1$. Differentiating (A 5), the left-hand side of (A 2) becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tfrac{2}{9}\{5-9s+2s(s+1)\}\,G_1(s)\,x^{-s}ds \tag{A 8}$$

and the right-hand side

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (-s)\,(-s-1)\,(-s-2)\,F_1(s)\,x^{-s-3}ds.$$

If $s(s+1)\,(s+2)\,F_1(s)$ is regular in $c-3 < \mathscr{R}s < c$, the contour of integration can be moved to $\mathscr{R}s = c-3$ in this integral. Then writing $s-3$ for s it may be rewritten

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (s-3)\,(s-2)\,(s-1)\,F_1(s-3)\,x^{-s}ds. \tag{A 9}$$

Equating the integrands of (A 8) and (A 9), and using (A 7), we get

$$\left(\frac{5-2s}{3}\right)\left(\frac{2-2s}{3}\right)G_1(s) = -(s-1)\,(s-2)\,(s-3)\,(\cot\tfrac{1}{2}\pi s)\,G_1(s-3). \tag{A 10}$$

Writing

$$G_1(s) = \left(-\frac{2s+1}{3}\right)!\,(s-1)!\,\phi(s), \tag{A 11}$$

the functional equation is simplified to

$$(\sin\tfrac{1}{2}\pi s)\,\phi(s) + (\cos\tfrac{1}{2}\pi s)\,\phi(s-3) = 0. \tag{A 12}$$

One solution is
$$\phi(s) = A \cos \tfrac{1}{2}\pi s, \tag{A 13}$$
where A is constant, and the possibility that this would be multiplied by some further function of period 3 is excluded by the same type of argument as used in § 3, since already $\phi(s) \sim \exp(-\tfrac{1}{3}\pi|\mathscr{I}s|)$ for large $|\mathscr{I}s|$, so this is the only solution. A is fixed from (A 3), since
$$\int_0^\infty g_1(\xi)\,d\xi = G(1) = A \lim_{s\to 1}\left(-\frac{2s+1}{3}\right)!\left(\cos\frac{\pi s}{2}\right) = -\tfrac{1}{2}.$$
The limit is $\tfrac{3}{4}\pi$, so $A = -2/3\pi$. Therefore
$$\left.\begin{matrix} F_1(s) \\ G_1(s)\end{matrix}\right\} = -\left(\frac{2}{3\pi}\right)\left(-\frac{2s+1}{3}\right)!\,(s-1)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{\pi s}{2} \tag{A 14}$$
and $s(s+1)(s+2)F_1(s)$ is regular as required in $c-3 < \mathscr{R}s < c$ if we choose $0 < c < 1$. The inversion formulae (A 5) can now be used to obtain series expansions for $f_1(x)$ and $g_1(x)$.

For $0 < x < 1$ the integrals are evaluated by moving the contour to the left over successive poles, namely those of $(s-1)!(\sin\tfrac{1}{2}\pi s) = 2\pi/(-s)!(\cos\tfrac{1}{2}\pi s)$ and of $(s-1)!(\cos\tfrac{1}{2}\pi s) = 2\pi/(-s)!(\sin\tfrac{1}{2}\pi s)$ at odd and even negative integers, whence
$$f_1(x) = -\frac{2}{3\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4n+1}{3}\right)!\frac{x^{2n+1}}{(2n+1)!} \tag{A 15}$$
and
$$g_1(x) = -\frac{2}{3\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4n-1!}{3}\right)!\frac{x^{2n}}{(2n)!}. \tag{A 16}$$
These series are absolutely convergent for $0 \leqslant x \leqslant 1$.

Asymptotic expansions are obtained as minus the sums of residues at the poles of $\{-(2s+1)/3\}!$, i.e. $s = 1, \tfrac{5}{2}, 4, \ldots$, to the right of $\mathscr{R}s = c$:
$$\left.\begin{matrix} f_1(x) \\ g_1(x)\end{matrix}\right\} \sim \frac{1}{\pi}\sum_{n=0}^{p}(-1)^n\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left(\frac{3n-1}{4}\pi\right)\frac{[\tfrac{1}{2}(3n-3)]!}{(n-1)!}x^{\frac{1}{2}(3n-1)} + o(x^p) \tag{A 17}$$
with the leading terms quoted in § 4.

In exactly the same way as above, the equations connecting the symmetric part f_2 of the downwash distribution and the antisymmetric part g_2 of the vorticity, namely
$$f_2(x) = -\frac{2}{\pi}\int_0^\infty\frac{\xi g_2(\xi)\,d\xi}{\xi^2-x^2} \tag{A 18}$$
together with (A 2) with changed suffix, can be solved to give
$$G_2(s) = B\left(-\frac{2s+1}{3}\right)!\,(s-1)!\left(\sin\frac{\pi s}{2}\right). \tag{A 19}$$
But since $\{-\tfrac{1}{3}(2s+1)\}!$ has a pole at $s = 1$, the other two factors being unity there, the requirement that $G_2(1) = \int_0^\infty g_2(\xi)\,d\xi$ should exist can only be satisfied with the trivial choice $B = 0$, i.e.
$$f_2(x) \equiv g_2(x) \equiv 0. \tag{A 20}$$